The Fractional Fourier Transform and The Linear Canonical Transform

Nu-Chuan Shen

The Fourier Transform

- Fourier Transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\,dt$$

- Inverses Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega t} F(\omega)\,d\omega$$

The Example of FT

- input is a even function

- input is a odd function

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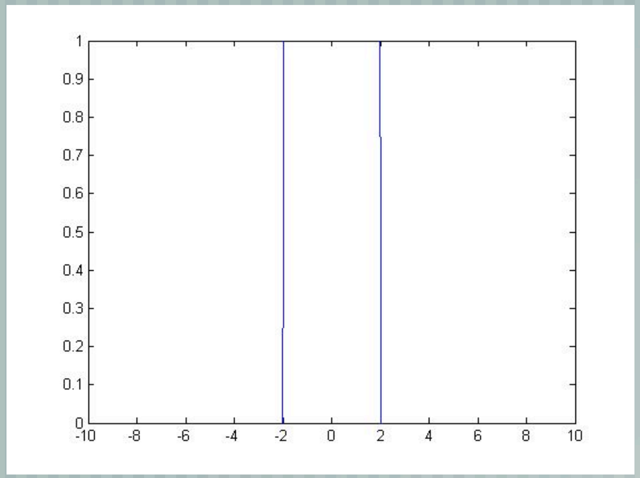

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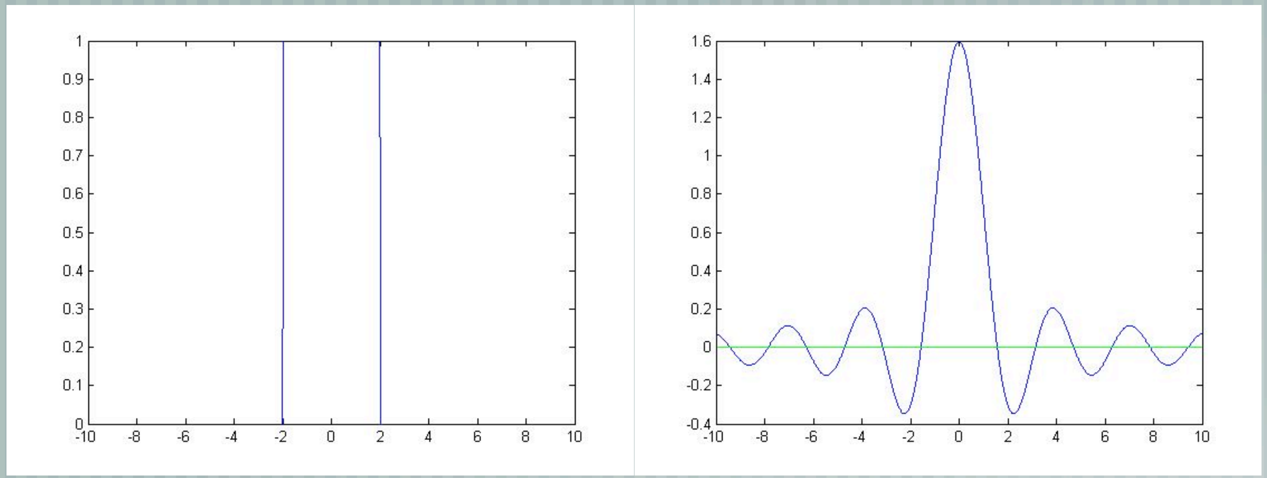

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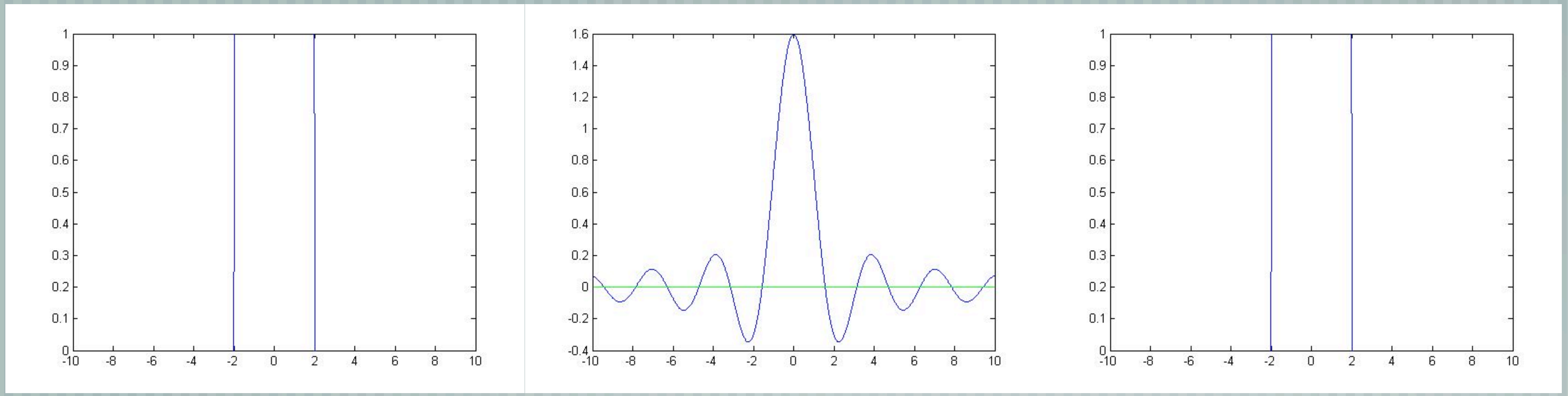

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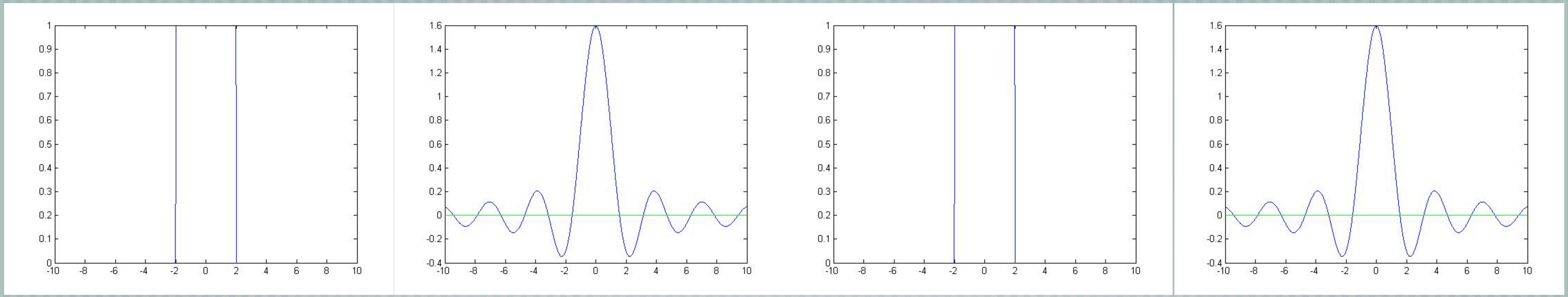

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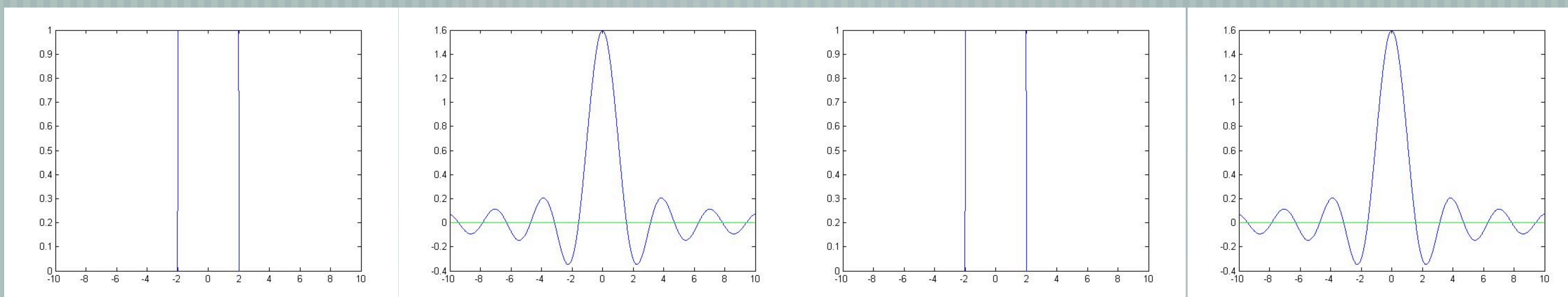

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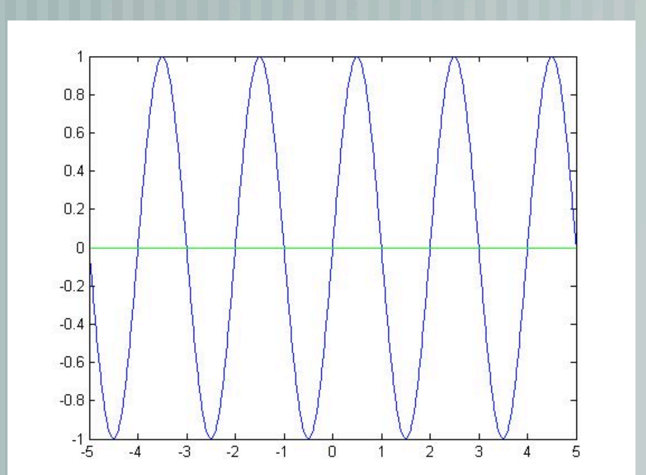

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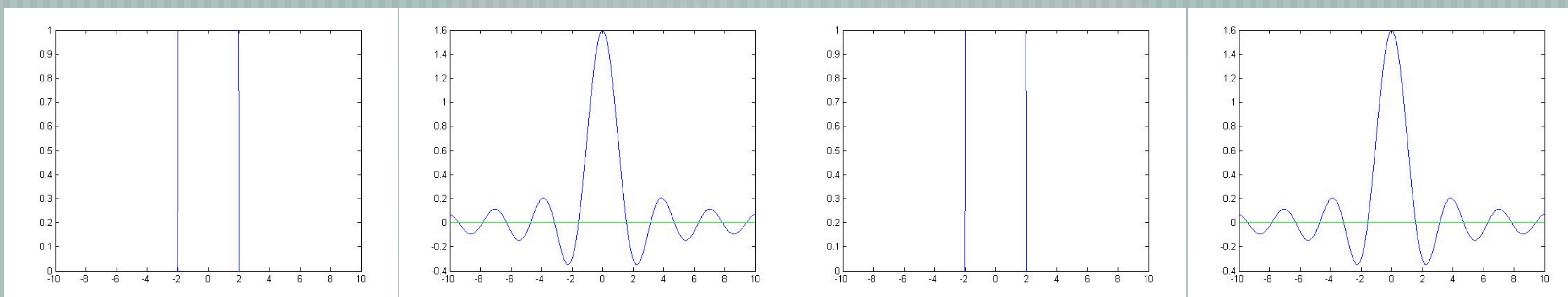

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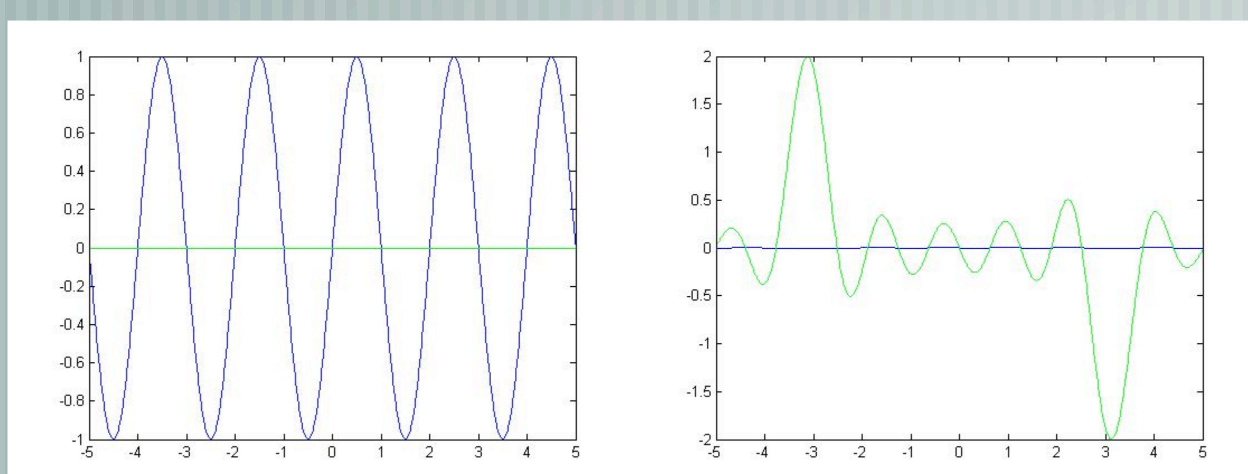

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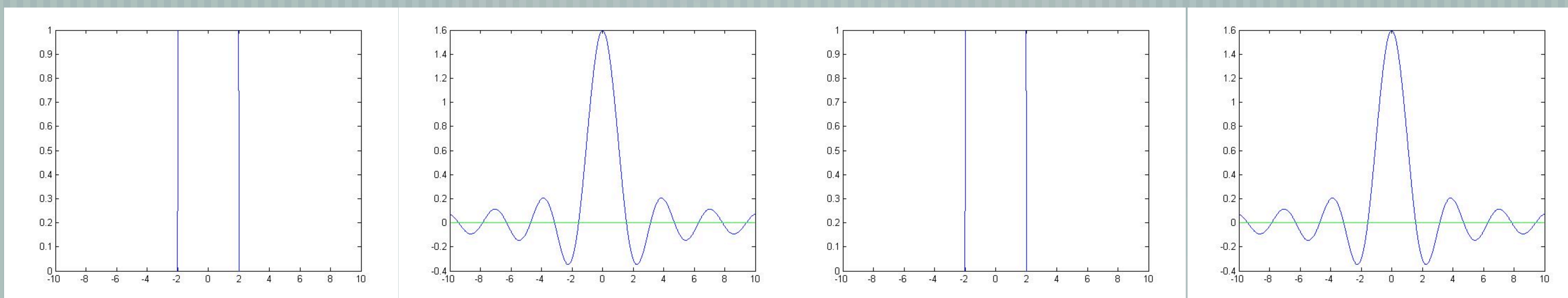

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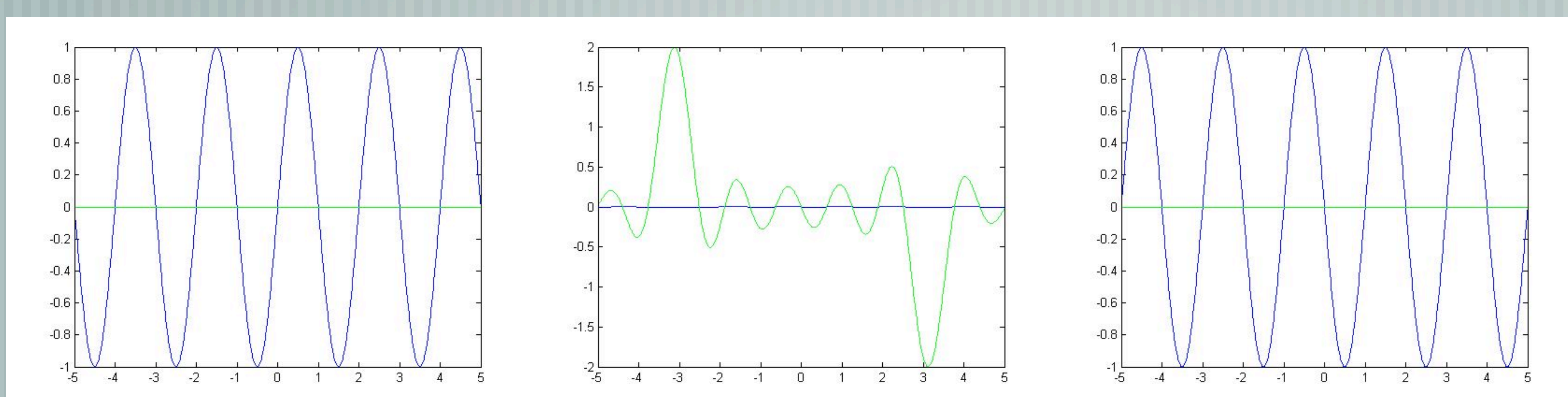

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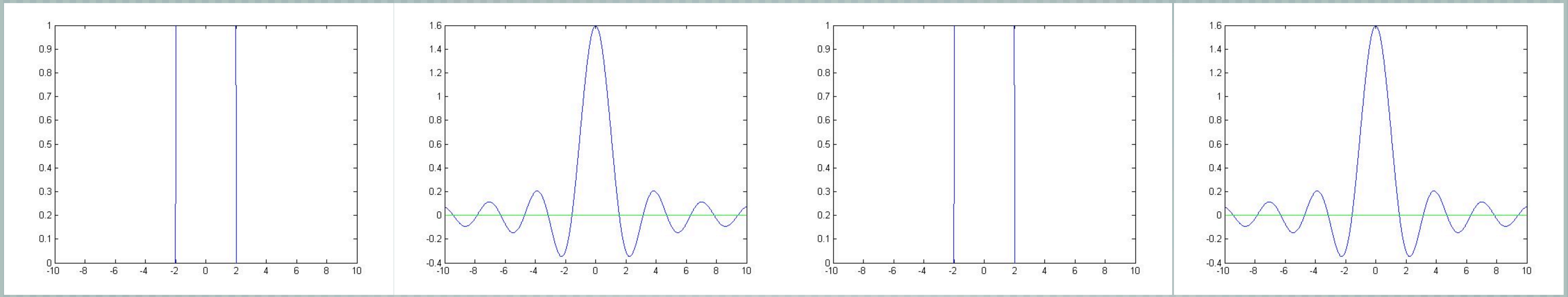

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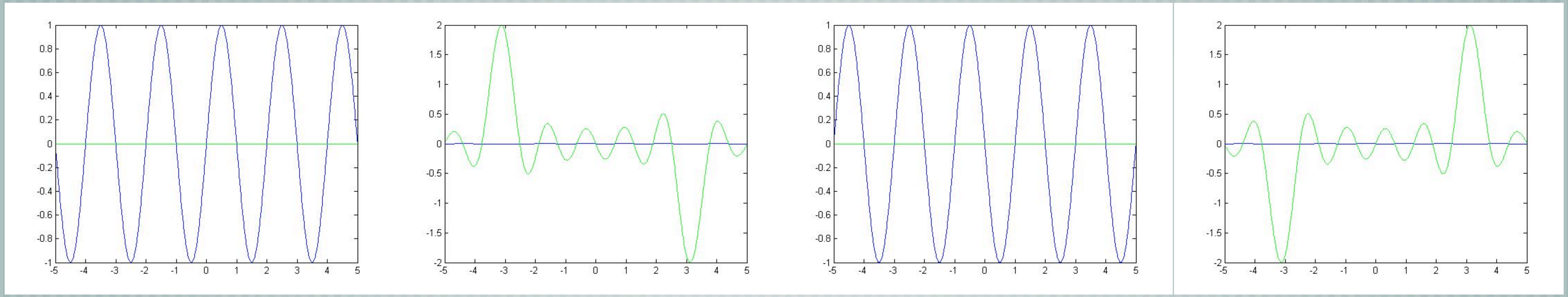


The Fractional Fourier Transform

- The definition of the fractional Fourier transform is:

$$O_F^{\alpha}(f(t)) = X_{\alpha}(u) = \int_{-\infty}^{\infty} K(\alpha,t,u)x(t)dt$$

, where the kernel is given by

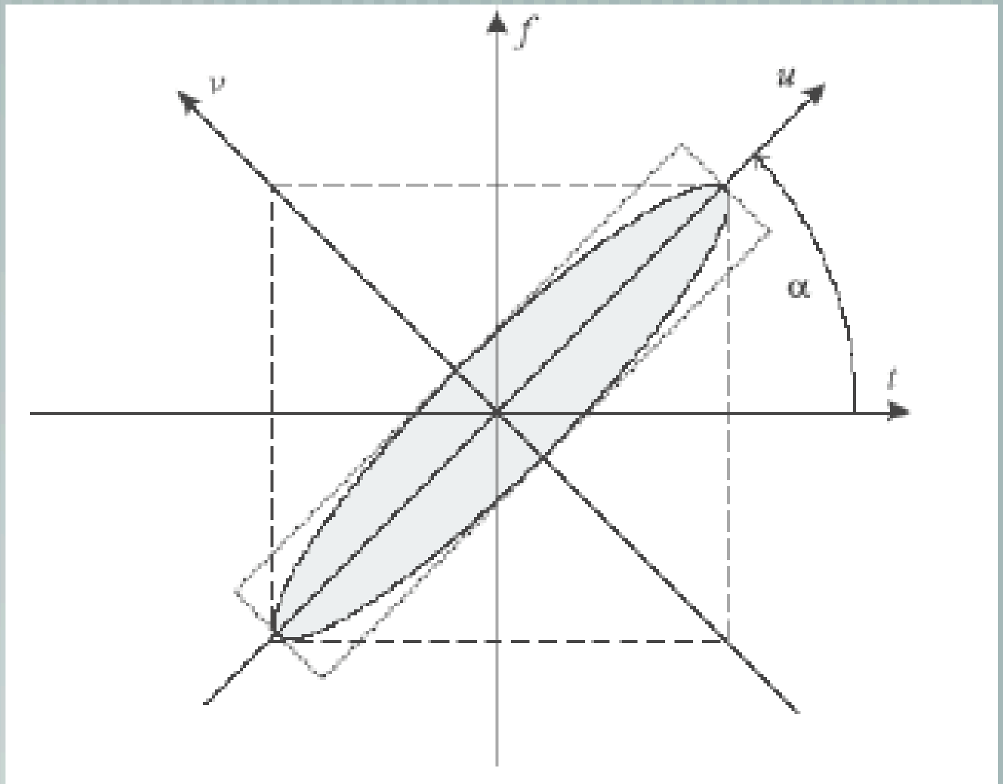
$$K(\alpha,t,u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}}\, e^{\frac{j}{2}\cot\alpha u^2}\, e^{-j\csc\alpha ut}\, e^{\frac{j}{2}\cot\alpha t^2}$$

$X_{\alpha}(u) = x(u)$ where $\alpha=2N\pi$ N is an integer

$X_{\alpha}(u) = x(-u)$ where $\alpha = (2N+1)\pi$ N is an integer

The Fractional Fourier domain

- In order to represent a signal in a new coordinate system, we use the rotation in the time-frequency plane by performing the fractional FT of the signal.



The Linear Canonical Transform

- The definition of the linear canonical transform is:

when b ≠ 0

$$O_F^{(a,b,c,d)}(f(t)) = F_{(a,b,c,d)}(u) = \sqrt{\frac{1}{j2\pi b}} e^{\frac{jd}{2b}u^2} \int_{-\infty}^{\infty} e^{-\frac{j}{b}ut} e^{\frac{ja}{2b}t^2} f(t)dt$$

when b=0

$$O_F^{(a,0,c,d)}(f(t)) = F_{(a,0,c,d)}(u) = \sqrt{d}e^{\frac{j}{2}cdu^2} f(du)$$

$$ad - bc = 1$$

The Freedom of The LCT with The FRFT

The FRFT:

$$O_F^{\alpha}(f(t)) = X_{\alpha}(u) = \int_{-\infty}^{\infty} K(\alpha,t,u)x(t)dt$$

The LCT:

when $b \neq 0$

$$O_F^{(a,b,c,d)}(f(t)) = F_{(a,b,c,d)}(u) = \sqrt{\frac{1}{j2\pi b}} e^{\frac{jd}{2b}u^2} \int_{-\infty}^{\infty} e^{-\frac{j}{b}ut} e^{\frac{ja}{2b}t^2} f(t)dt$$

	FRFT	LCT
number of the variant	1	4
freedom of transform	1	3

The additivity property of the LCT

- The additivity property of the LCT

$$O_F^{(a_2,b_2,c_2,d_2)}(O_F^{(a_1,b_1,c_1,d_1)}(f(t))) = O_F^{(e,f,g,h)}(f(t))$$

, where the (e, f, g, h) is

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

The Inverse LCT

- According to the additivity property, the inverse LCT is defined as:

$$O_F^{(d,-b,-c,a)}(O_F^{(a,b,c,d)}(f(t))) = f(t)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \because ad - bc = 1$$

The Special Case of LCT (I)

- Case 1 the (a, b, c, d) = (0, 1, -1, 0)

when $b \neq 0$

$$O_F^{(a,b,c,d)}(f(t)) = F_{(a,b,c,d)}(u) = \sqrt{\frac{1}{j2\pi b}} e^{\frac{jd}{2b}u^2} \int_{-\infty}^{\infty} e^{-\frac{j}{b}ut} e^{\frac{ja}{2b}t^2} f(t)dt$$

$$\rightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt$$

$$O_F^{(0,1,-1,0)}(f(t)) = \sqrt{-j} FT(f(t))$$

The Special Case of LCT (II)

- Case 2 the $(a, b, c, d) = (0, -1, 1, 0)$

$$O_F^{(0,-1,1,0)}(F(\omega)) = \sqrt{j}\, IFT(F(\omega))$$

The Special Case of LCT (III)

- Case 3 the (a, b, c, d) = (cosα, sinα, -sinα, cosα)

$$O_F^{(\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)}(f(t)) = (e^{-j\alpha})^{1/2} O_F^{\alpha}(f(t))$$

The Special Case of LCT (IV)

- Case 4 the $(a, b, c, d) = (1, 0, \tau, 1)$

$$O_F^{(\alpha,0,c,d)}(f(t)) = e^{\frac{j}{2}\tau u^2} f(t)$$

The Special Case of LCT (V)

- Case 5 the (a, b, c, d) = (σ, 0, 0, 1/σ)

$$O_F^{(\sigma,0,0,\sigma^{-1})}(f(t)) = \sqrt{\sigma^{-1}}\, e^{\frac{j}{2\sigma}u^2} f(\sigma^{-1}t)$$

WHY??? WHY???

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- Why we need to discuss the fractional FT moment !?

WHY??? WHY???

- Why we need to discuss the fractional FT moment !?
- It can help us find out the extreme width of the signal in the fractional fourier domain.

Ambiguity Function

- The definition of the ambiguity function is:

$$A_f(t,w) = \int_{-\infty}^{\infty} x(\tau + t/2)x^*(\tau - t/2)\exp(-j2\pi wt)d\tau$$

Fractional Fourier Transform Moments

- The fractional FT corresponds to a rotation of the AF

$$t = R\cos\alpha \qquad w = R\sin\alpha \qquad R \in [-\infty, \infty] \qquad \alpha \in [0, \pi)$$

- The relationship between the AF in this coordinate system

$$\tilde{A}_f(R, \alpha - \pi/2) = \int_{-\infty}^{\infty} |X_\alpha(t)|^2 \exp(j2\pi Rt)dt$$

Fractional Fourier Transform Moments

- The zero order moment

$$E = \int_{-\infty}^{\infty} \left| X_\alpha(t) \right|^2 dt = \tilde{A}_f\,(R, \alpha - \pi / 2)\Big|_{R=0} = A_f(0,0)$$

Fractional Fourier Transform Moments

- The first order moments

$$m_\alpha = \int_{-\infty}^{\infty} \left|X_\alpha(t)\right|^2 t dt = \frac{1}{E}\frac{1}{2\pi j}\left.\frac{\partial \tilde{A}_f(R,\alpha-\pi/2)}{\partial R}\right|_{R=0}$$

$$\left.\frac{\partial \tilde{A}_f(R,\alpha-\pi/2)}{\partial R}\right|_{R=0,\alpha=\pi/2} = \left.\frac{\partial A_f(t,w)}{\partial t}\right|_{t=0,w=0} = 2\pi j\int_{-\infty}^{\infty}\left|X_{\pi/2}(w)\right|^2 w dw$$

$$\left.\frac{\partial \tilde{A}_f(R,\alpha-\pi/2)}{\partial R}\right|_{R=0,\alpha=\pi} = \left.\frac{\partial A_f(t,w)}{\partial w}\right|_{t=0,w=0} = 2\pi j\int_{-\infty}^{\infty}\left|x(-t)\right|^2 t dt$$

- rewrite it in a generalization of two special case

$$m_\alpha = m_0\cos\alpha + m_{\pi/2}\sin\alpha$$

Fractional Fourier Transform Moments

- The second order moments is defined as:

$$\omega_\alpha = \frac{1}{E}\int_{-\infty}^{\infty} |X_\alpha(t)|^2 t^2 dt = \frac{1}{E}\left(\frac{1}{j2\pi}\right)^2 \frac{\partial^2 \tilde{A}_f(R, \alpha - \pi/2)}{\partial R^2}\Bigg|_{R=0}$$

- The second order central moments is defined as:

$$P_{\alpha =} \frac{1}{E}\int_{-\infty}^{\infty} |X_\alpha(t)|^2 (t - m_\alpha)^2 dt = (\omega_\alpha - m_\alpha{}^2)$$

Fractional Fourier Transform Moments

- The second order moments can be rewritten as:

$$\omega_\alpha = \omega_0 \cos^2\alpha + \omega_{\pi/2}\sin^2\alpha + \left[\omega_{\pi/4} - (\omega_0 + \omega_{\pi/2})/2\right]\sin 2\alpha$$

- The second order central moments can be rewritten as:

$$p_\alpha = p_0 \cos^2\alpha + p_{\pi/2}\sin^2\alpha + \left[\omega_{\pi/4} - m_0 m_{\pi/2} - (\omega_0 + \omega_{\pi/2})/2\right]\sin 2\alpha$$

Fractional Fourier Transform Moments

- first derivative of the second-order central FRFT

$$\frac{dp_\alpha}{d\alpha} = \left(p_{\pi/2} - p_0\right)\sin 2\alpha + \left[2\left(\omega_{\pi/4} - m_0 m_{\pi/2}\right) - \left(\omega_0 + \omega_{\pi/2}\right)\right]\cos 2\alpha = 0$$

- Optimal rotation angle

$$\tan 2\alpha_e = \frac{2\left(\omega_{\pi/4} - m_0 m_{\pi/2}\right) - \left(\omega_0 + \omega_{\pi/2}\right)}{\left(p_0 - p_{\pi/2}\right)}$$

Time-Frequency Analysis

- Short Time Fourier Transform
- Garbor Transform
- Wigner distribution function
- Pseudo Wigner distribution function
- S-method Transfomr

WHAT !? WHAT!?

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- another drawback of the FT !?

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- another drawback of the FT !?
- We can not judge the instant frequency of the signal.

WHAT !! WHAT!!

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- Is there any relationship between the FRFT and LCT with time-frequency analysis?

WHAT !! WHAT!!

- Is there any relationship between the FRFT and LCT with time-frequency analysis?
- The most important property of the FTFT and LCT --rotation property can be observed by the time-frequency analysis.

Short Time Fourier Transform

- Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\,dt$$

- Short time fourier transform

$$ST_x(t, f) = \int_{-\infty}^{\infty} x(t + t_0) g^*(t_0) \exp(-j2\pi t_0 f)\,dt_0$$

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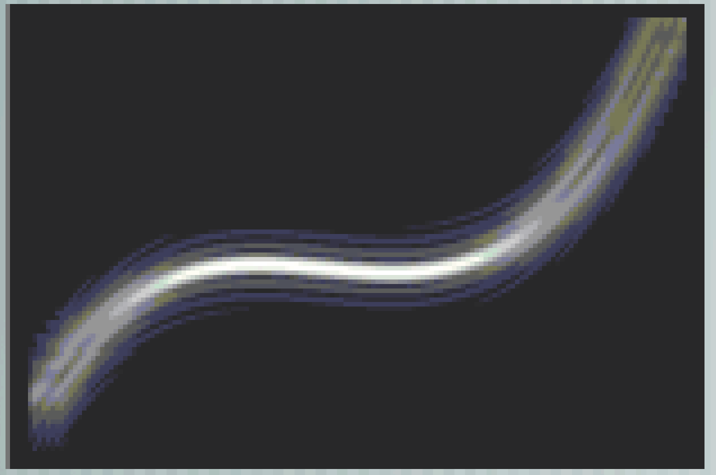

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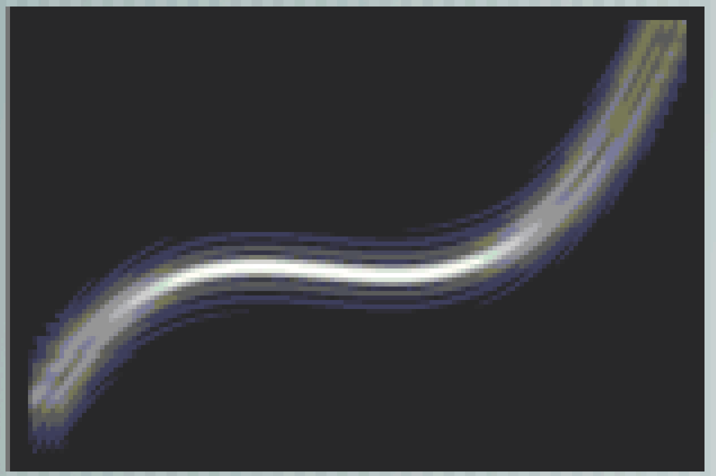

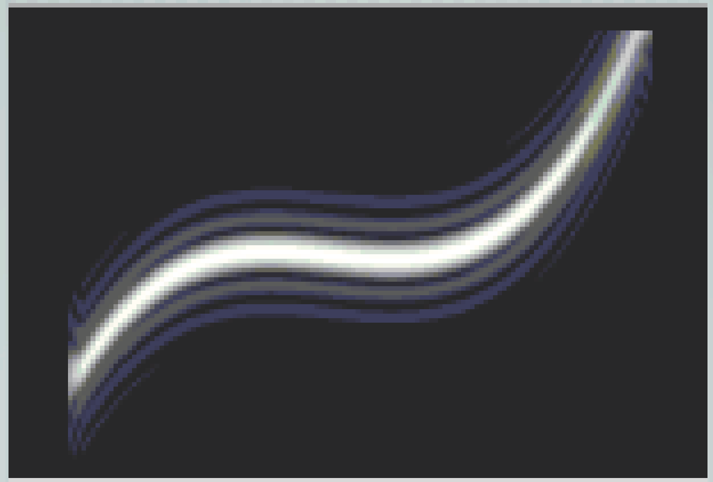

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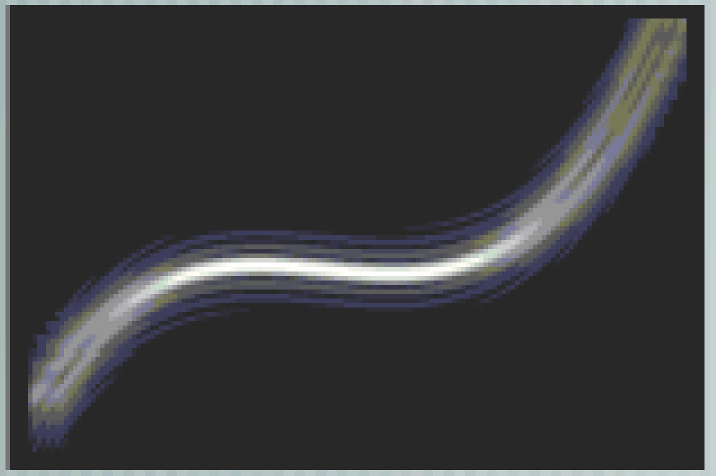
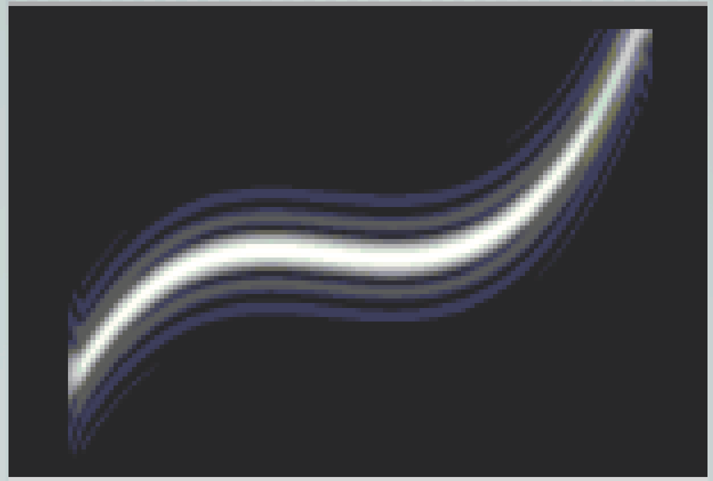
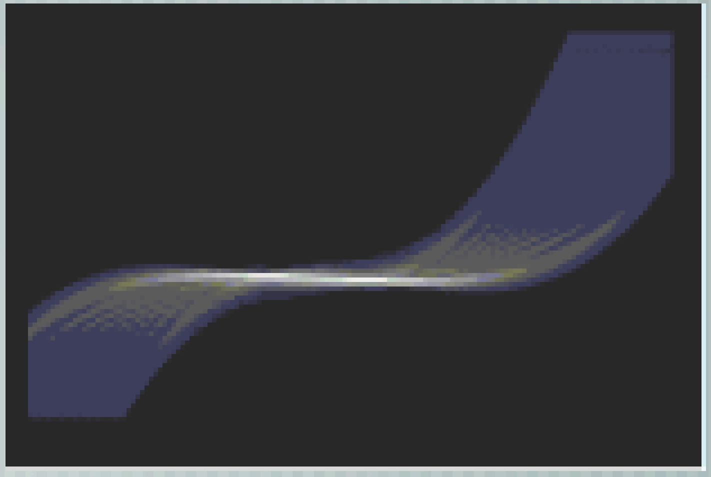




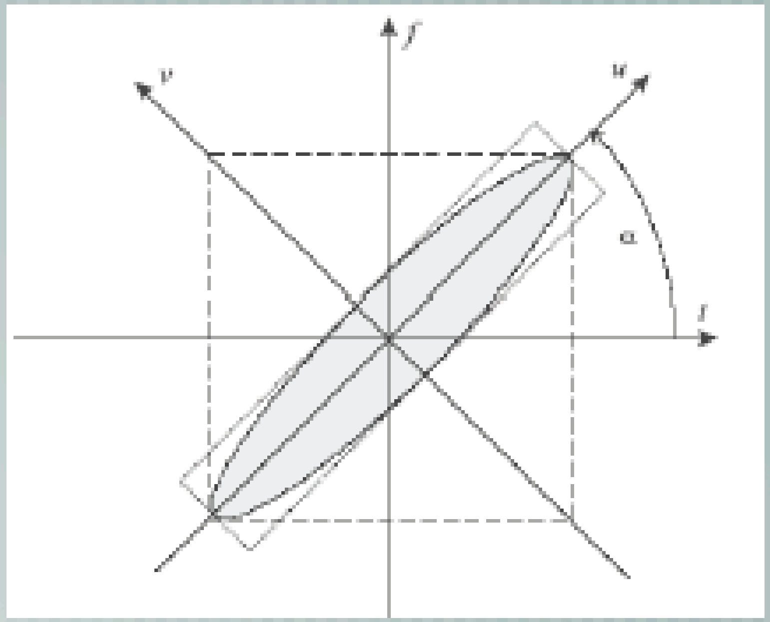
Short Time Fourier Transform

- The STFT of the signal in the fractional FT domain is defined as:

$$ST_x^{\alpha}(u,v) = \int_{-\infty}^{\infty} X_{\alpha}(u+u_0)g^*(u_0)\exp(-j2\pi u_0 v)du_0$$

- The rotation relationship is

$$\begin{pmatrix} t \\ f \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$



Wigner Distribution Function

- The WDF of a signal *x(t)* is defined as

$$W_f(t,w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-jw\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w+\eta/2)X^*(w-\eta/2)\mathrm{e}^{jt\eta}d\eta$$

- There is a big problem of the WDF — cross-term

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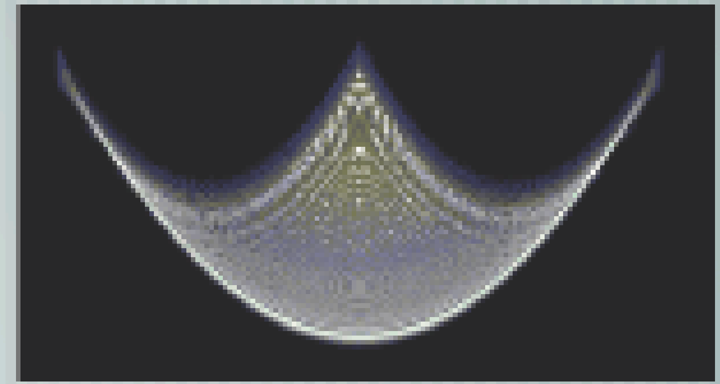

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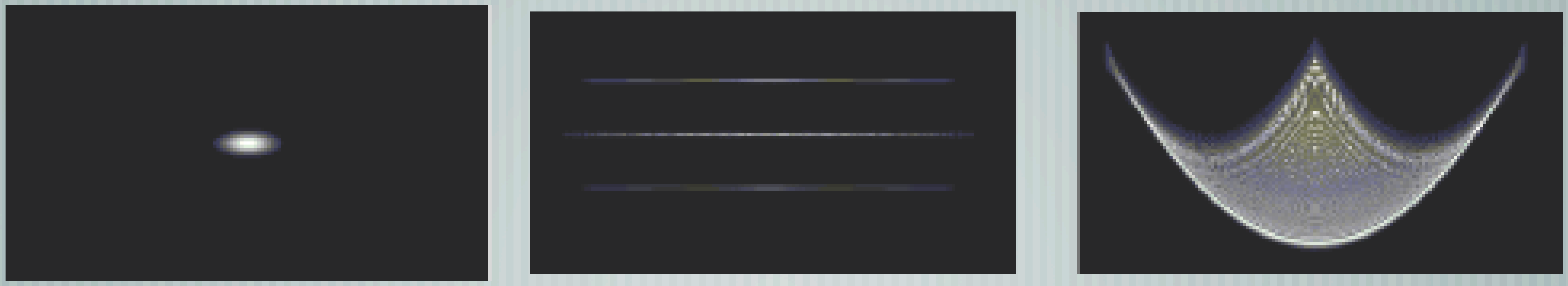

Wigner Distribution Function

The relations between WDF and FRFT

$$W_{F_\alpha}(u,v) = W_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)$$

The relations between WDF and LCT

$$W_{F_{(a,b,c,d)}}(u,v) = W_f(du - bv, -cu + av)$$

$$W_{F_{(a,b,c,d)}}(au + bv, cu + dv) = W_f(u,v)$$

Time-Frequency Analysis

Time-Frequency Analysis

- The advantage of the STFT
  no cross-term problem

Time-Frequency Analysis

- The advantage of the STFT
  no cross-term problem
- the disadvantage of the STFT
  the resolution is low

Time-Frequency Analysis

Time-Frequency Analysis

- The advantage of the WDF
  the resolution is high

Time-Frequency Analysis

- The advantage of the WDF
  the resolution is high
- The disadvantage of the WDF
  cross-term problem

Pseudo WDF

- Pseudo Wigner Distribution Function

$$PWD_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)g^*(\tau/2)g(-\tau/2)\exp(-j2\pi\tau f)d\tau$$

- Short Time Fourier Transform

$$ST_x(t,f)=\int_{-\infty}^{\infty} x(t+t_0)g^*(t_0)\exp(-j2\pi t_0 f)dt_0$$

- Wigner Distribution Function

$$W_f(t,w)=\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-jw\tau}d\tau$$

Pseudo WDF

- The pseudo WDF can also be expressed in terms of the STFT as:

$$PWD_x(t,f) = \int_{-\infty}^{\infty} ST_x(t, f + \theta/2) ST_x^*(t, f - \theta/2) d\theta$$

Pseudo WDF

- Expand the pseudo WDF

$$x(t) = \sum_{i=1}^{M} x_i(t)$$

$$PWD_x(t,f) = \int_{-\infty}^{\infty} ST_x(t, f+\theta/2) ST_x^*(t, f-\theta/2) d\theta$$

$$PWD_x(t,f) = \sum_{i=1}^{M} PWD_{x_i}(t,f) \qquad (auto-terms)$$

$$+ \sum_{i=1}^{M} \sum_{k=1 k\neq i}^{M} PWD_{x_i, x_k}(t,f) \qquad (cross-terms)$$

S-method

- Based on the definition of the pseudo WDF, the S-method for time frequency analysis can be written as:
- on frequency-direction combined STFT

$$P_x(t,f) = \int_{-\infty}^{\infty} ST_x(t, f+\theta/2) z(\theta) ST_x^*(t, f-\theta/2) d\theta$$

- on time-direction combined STFT

$$P_x(t,f) = \int_{-\infty}^{\infty} ST_x(t+\theta/2, f) z(\theta) ST_x^*(t-\theta/2, f) \exp(-j2\pi f\theta) d\theta$$

S-method

- The S-method in this fractional domain is

$$P_x(t,f)=\int_{-\infty}^{\infty} ST_x^{\alpha}(u,v+\theta/2)z(\theta)ST_x^{a^*}(u,v-\theta/2)d\theta$$

S-method

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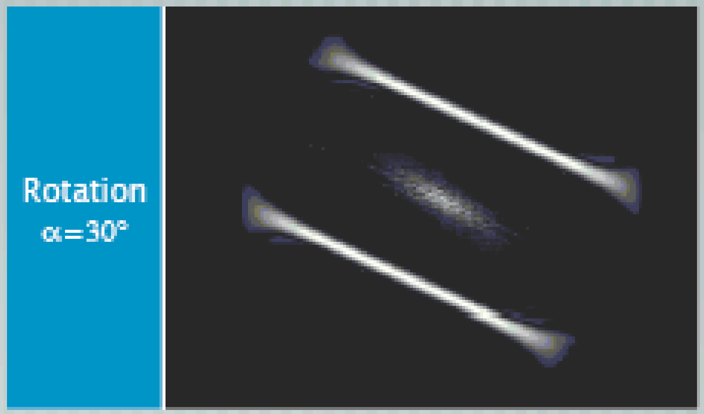

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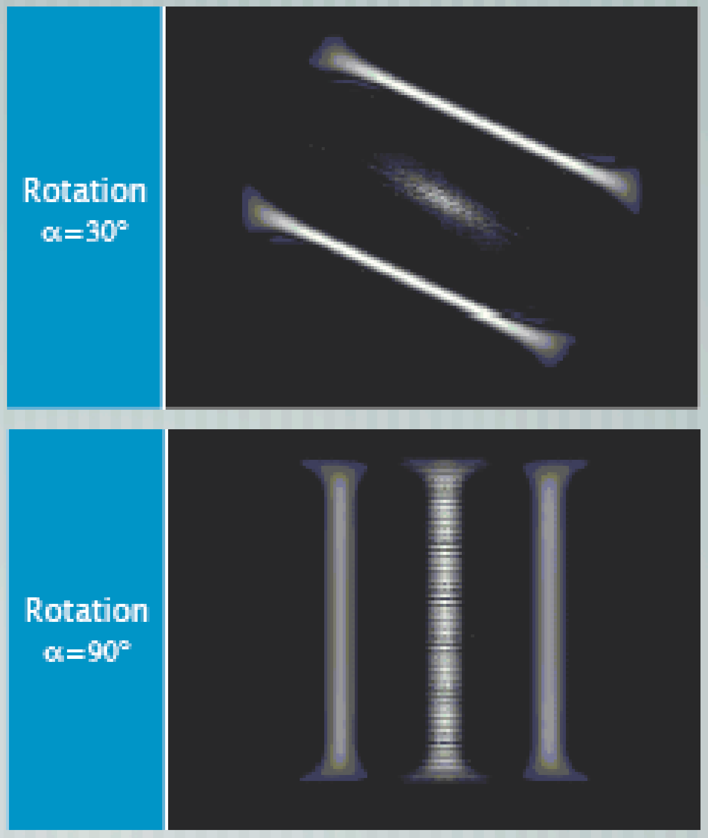

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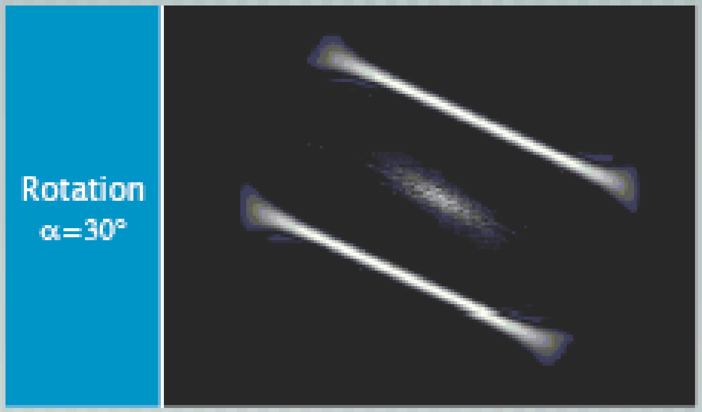
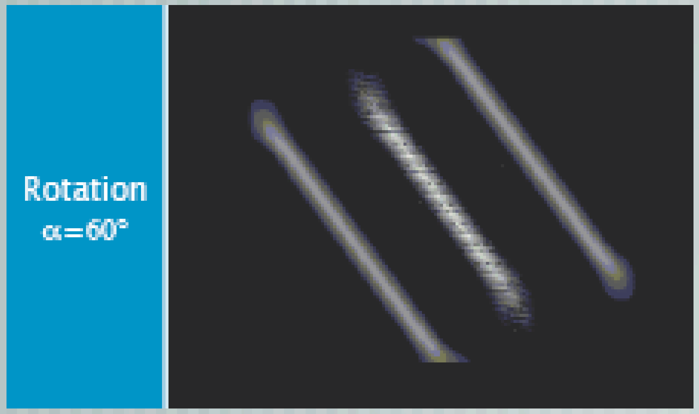
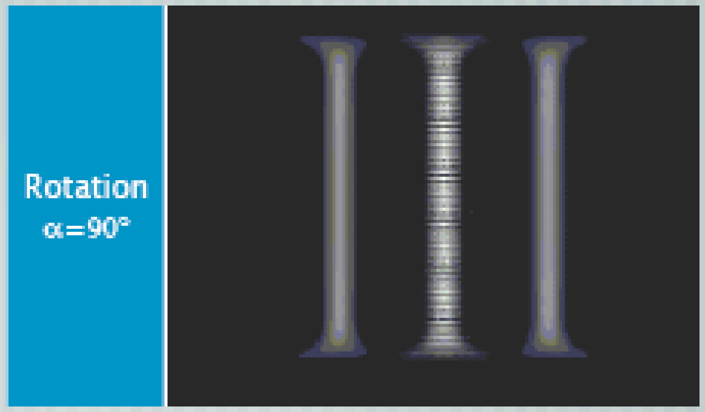


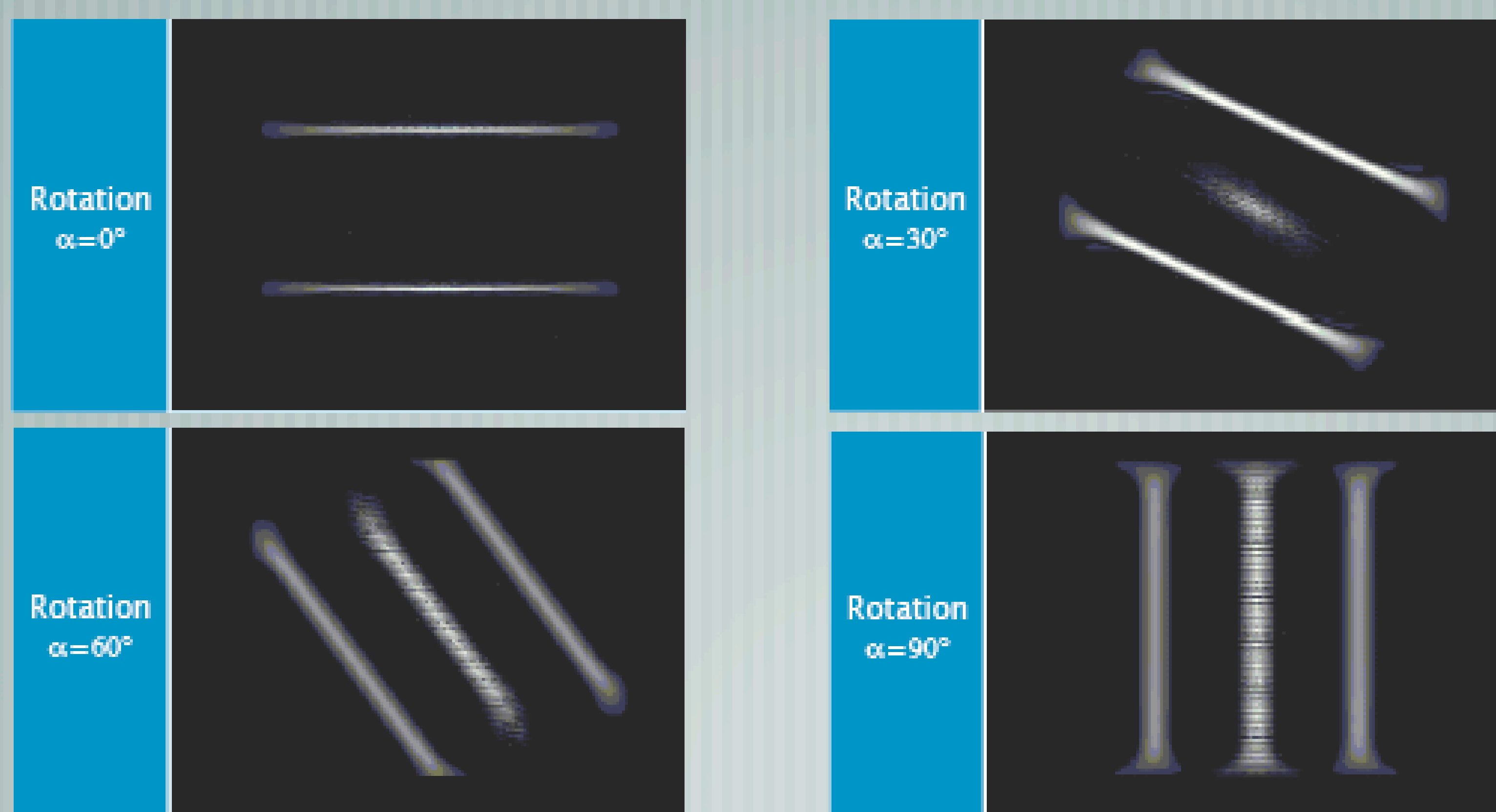




S-method

- The S-method in this fractional domain is

$$P_x(t,f) = \int_{-\infty}^{\infty} ST_x^{\alpha}(u, v+\theta/2)\, z(\theta)\, ST_x^{a^*}(u, v-\theta/2)\, d\theta$$



Comparison

- comparison with the performance of STFT, WDF, S-method

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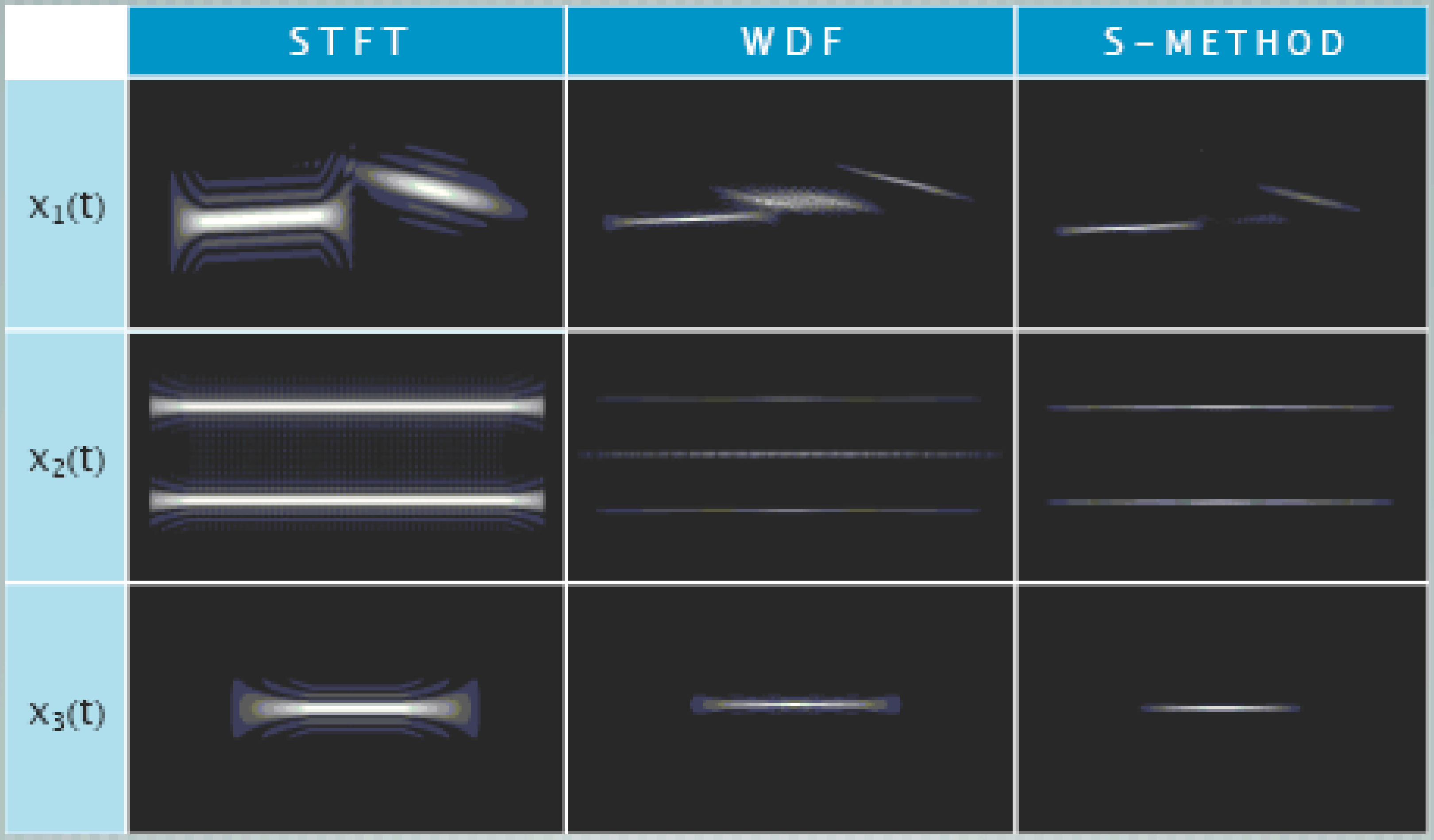

Filter Design part 1

- Optimal canonical filter:

Filter Design part 1

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$R_{si}(t,\sigma)$ is the cross-correlation between the original signal s(t) and the received signal $x_i(t)$.

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Filter Design part 2

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- 2. how to determine the cutoff criteria.

Filter Design part 2

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- STEP2 : Separating the signal and noise component by determining the cutoff lines on the t-w plane of the S-method.
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- STEP4 : Filtering the noises by passing it through the filter with the parameter in step3.

Filter Design part 2

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$$f_1(t) = O_F^{-\alpha_1}\left\{O_F^{\alpha_1}\left[f(t)\right]H_1(u)\right\},$$

$$f_2(t) = O_F^{-\alpha_2}\left\{O_F^{\alpha_2}\left[f_1(t)\right]H_2(u)\right\},$$

$$\vdots$$

$$f_{n-1}(t) = O_F^{-\alpha_{n-1}}\left\{O_F^{\alpha_{n-1}}\left[f_{n-2}(t)\right]H_{n-1}(u)\right\},$$

$$r(t) = O_F^{-\alpha_n}\left\{O_F^{\alpha_n}\left[f_{n-1}(t)\right]H_n(u)\right\}$$

Filter Design part 2

example

$$s(t) = 2\cos(5t)e^{(-t^2/10)}$$

$$n(t) = 0.5e^{j0.23t^2} + 0.5e^{j0.3t^2 + j8.5t} + 0.5e^{j0.46t^2 - j9.6t}$$

SIGNAL: s(t) without noise
ANALYSIS: t-domain

SIGNAL: s(t) with noise
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Filter Design part 2

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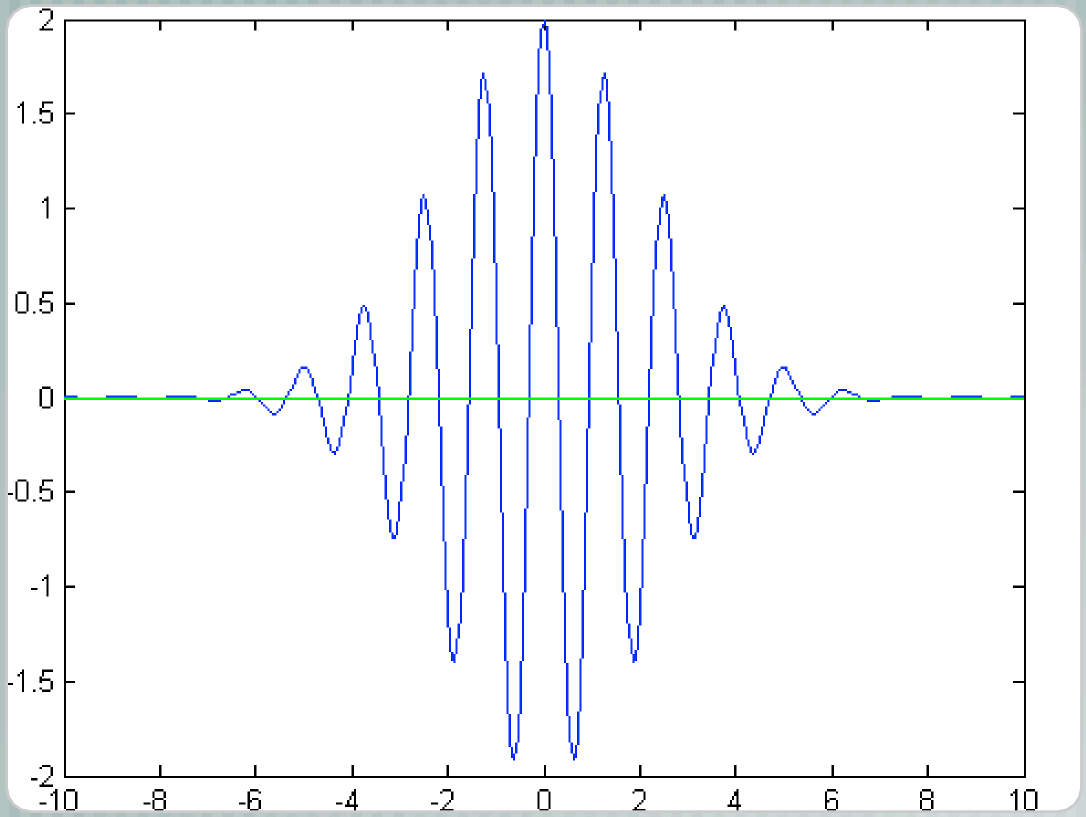

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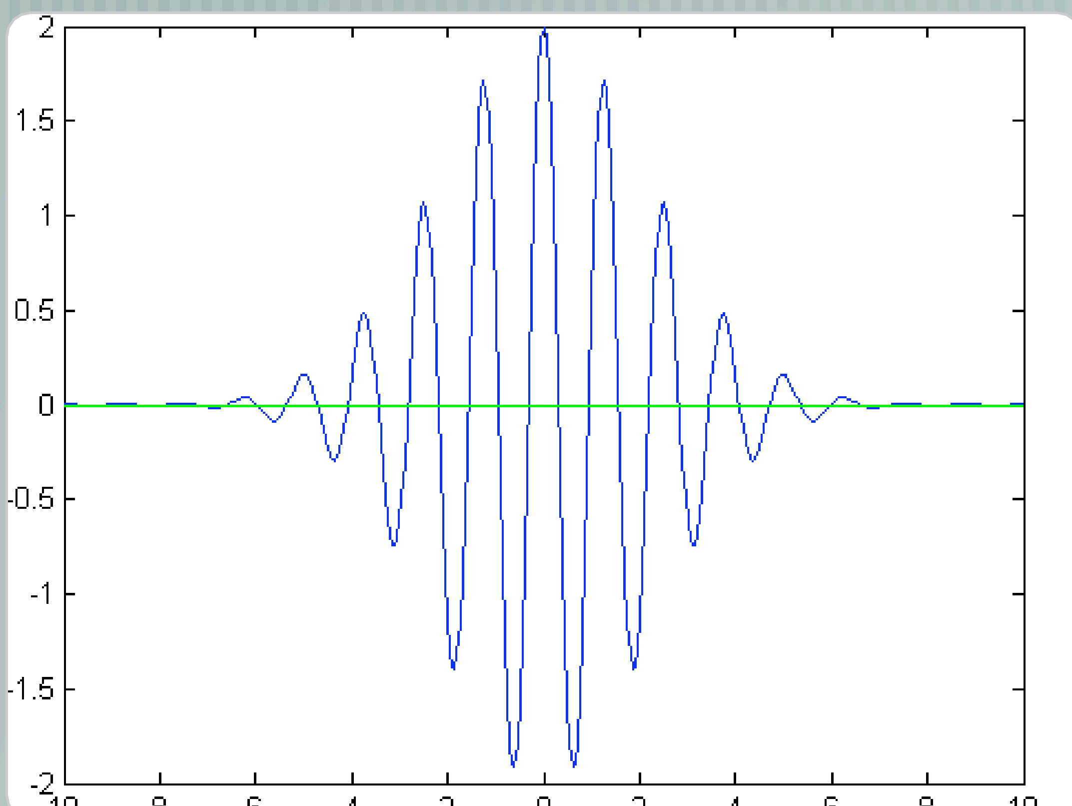
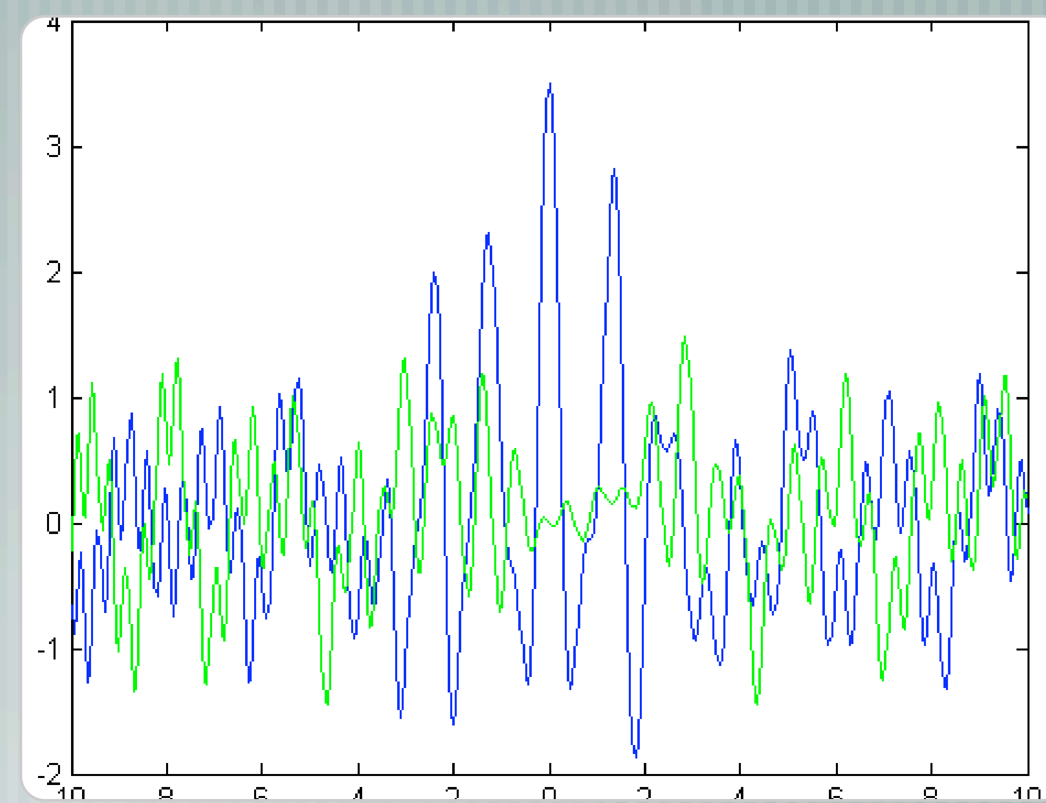


SIGNAL: s(t) without noise

ANALYSIS: t-domain



SIGNAL: s(t) with noise

ANALYSIS: t-domain

Filter Design part 2

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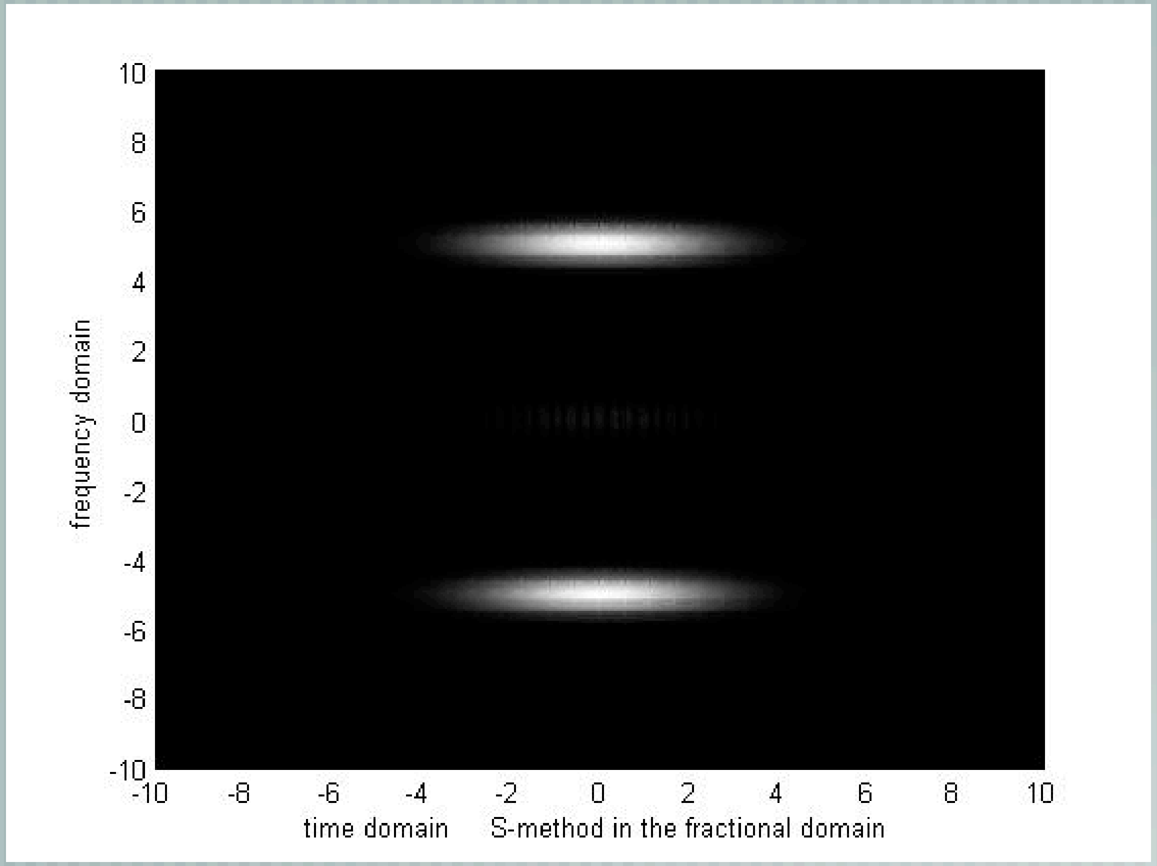
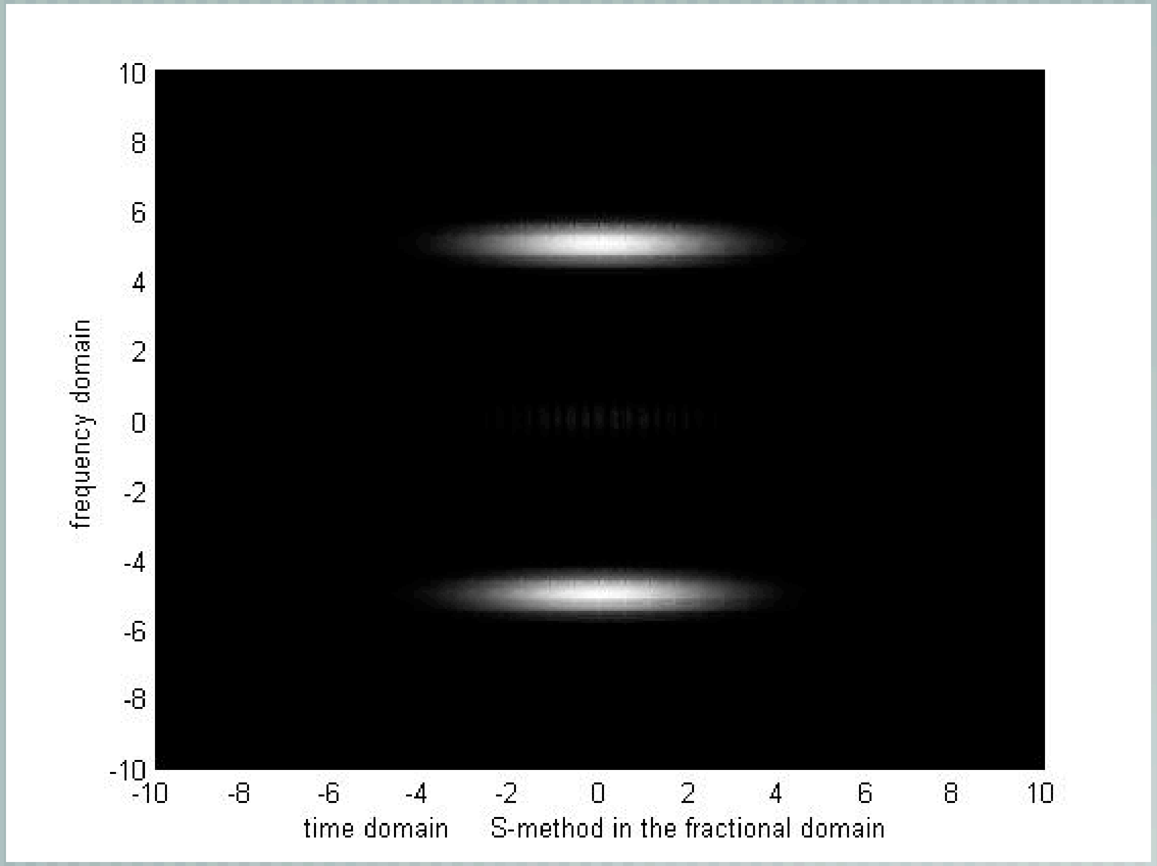


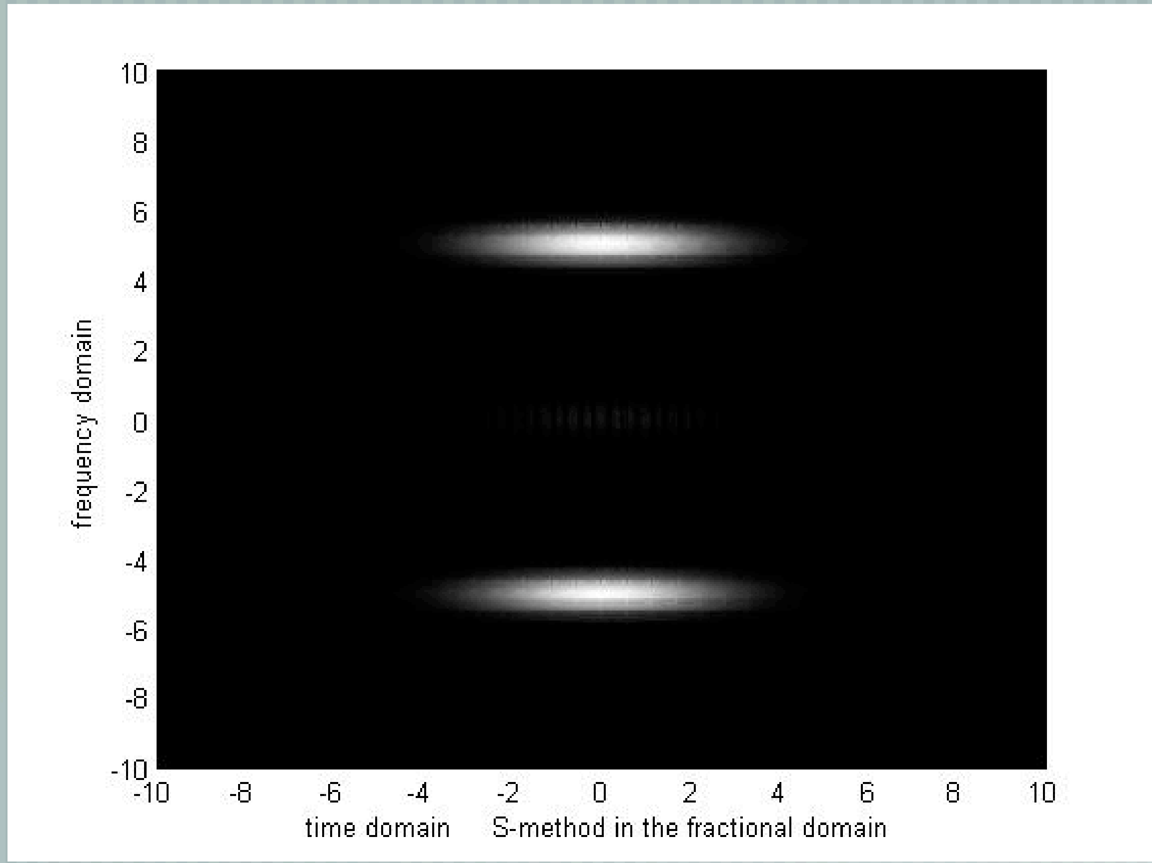
Filter Design part 2

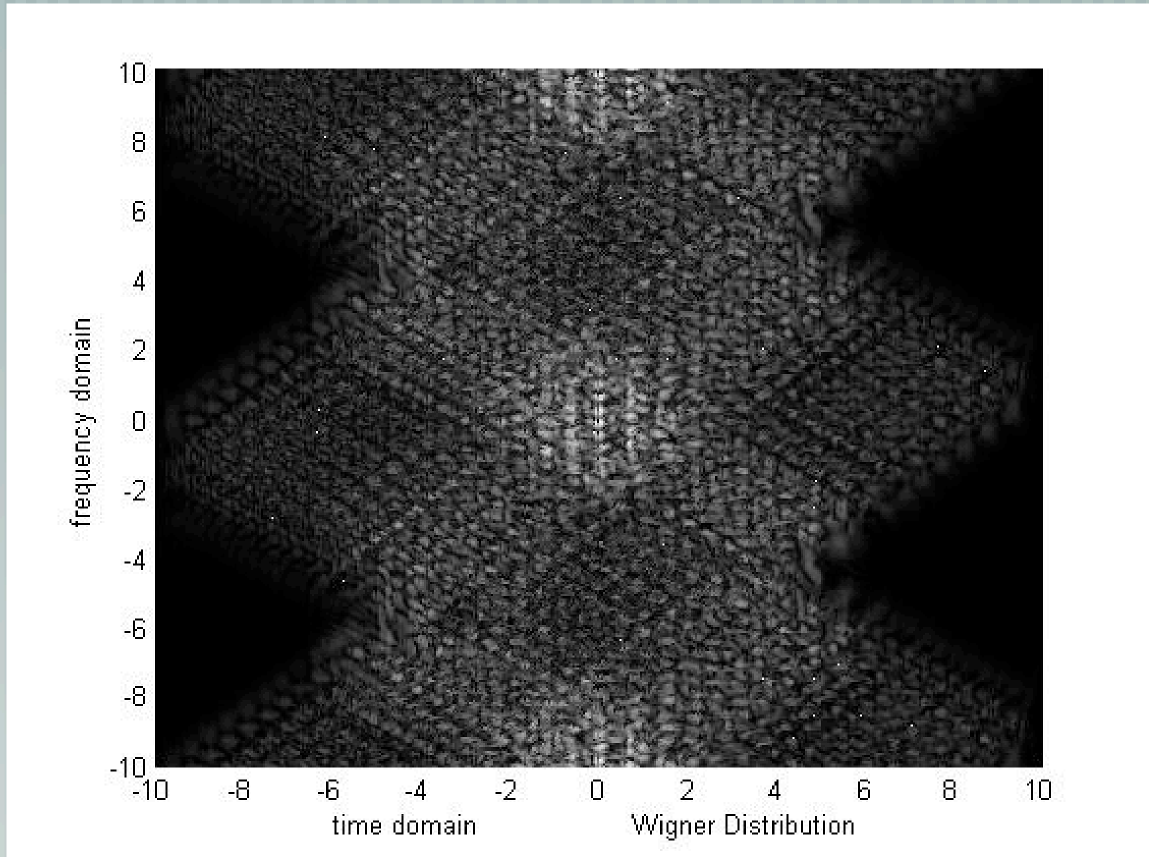


SIGNAL: s(t) with S-method

ANALYSIS: t-domain

Filter Design part 2

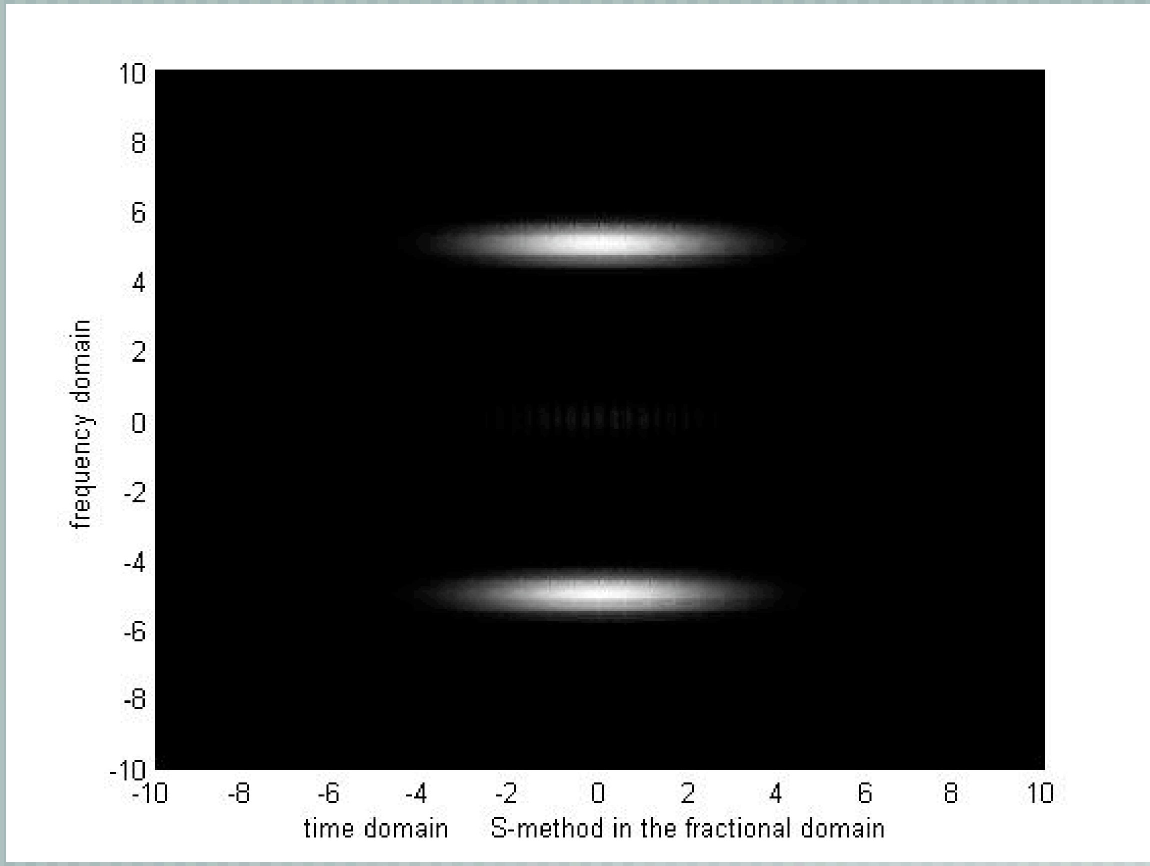


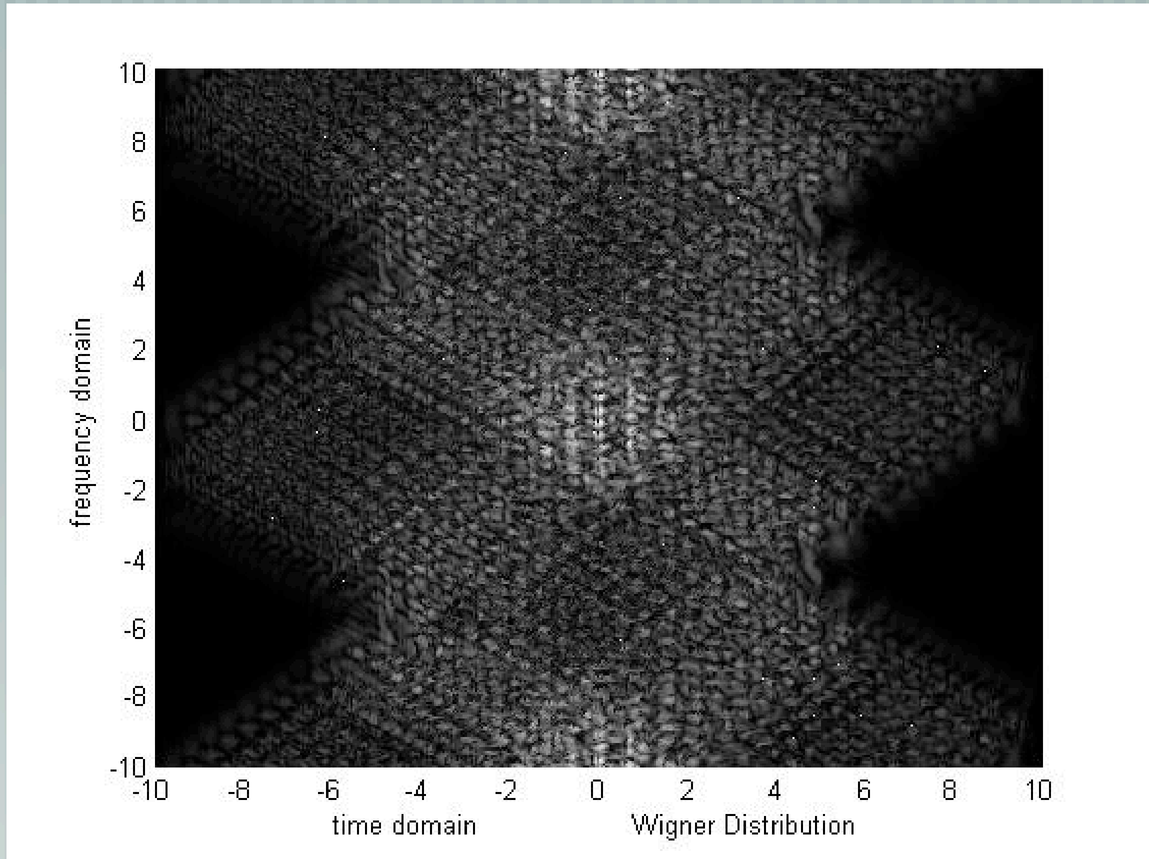


SIGNAL: s(t) with S-method

ANALYSIS: t-domain

Filter Design part 2





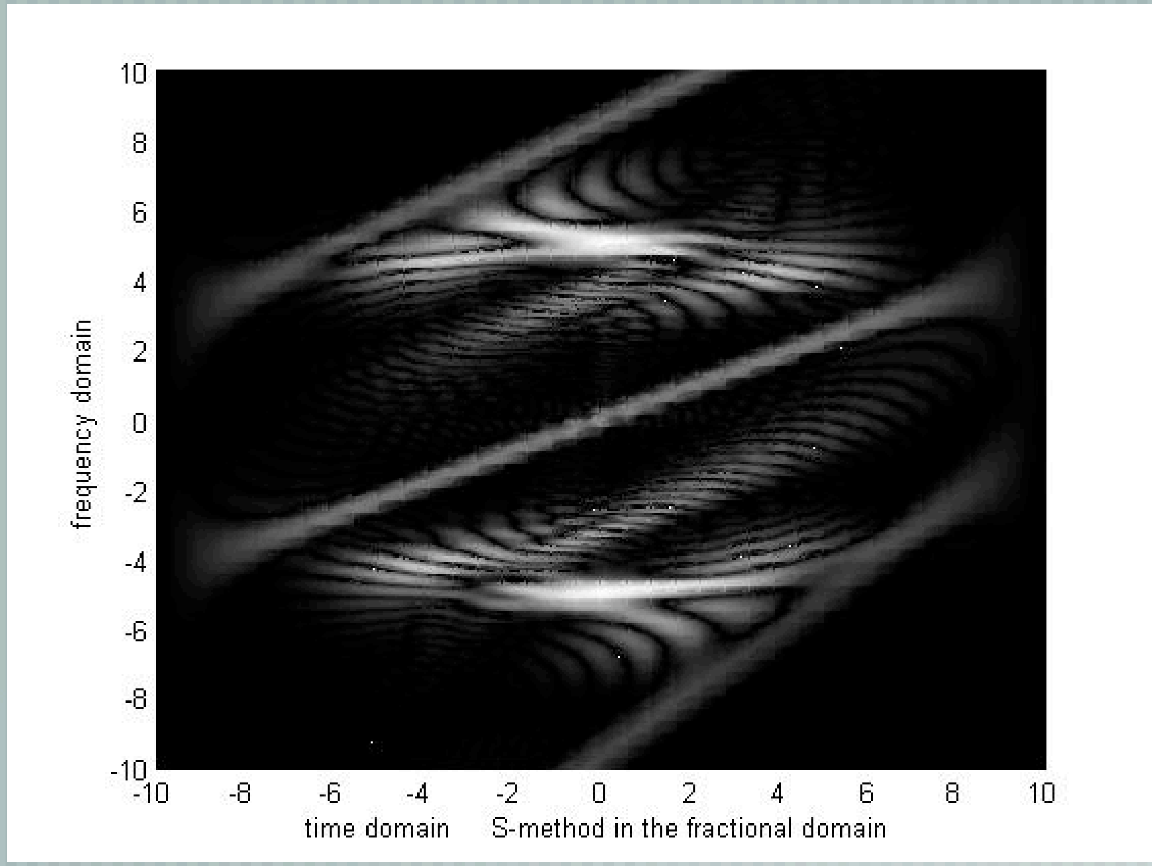
SIGNAL: s(t) with S-method

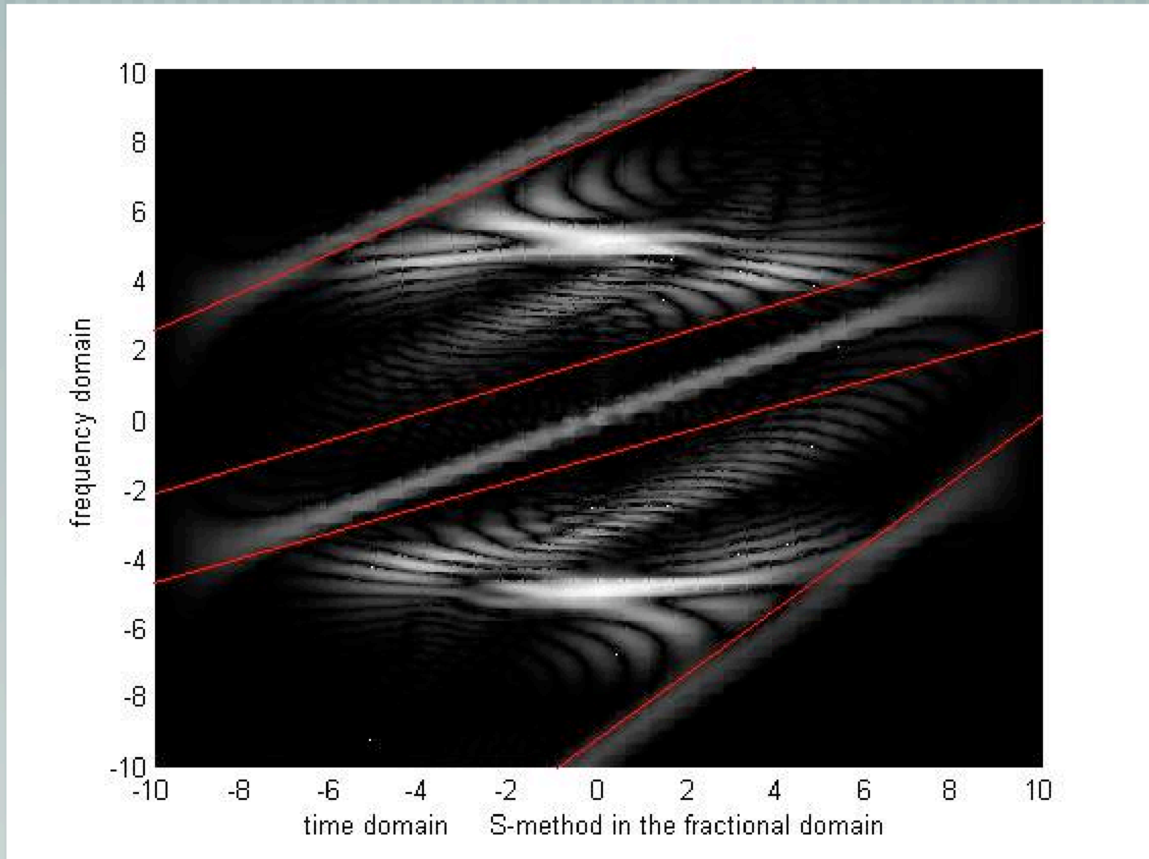
ANALYSIS: t-domain

SIGNAL: s(t) with WDF

ANALYSIS: t-domain

Filter Design part 2





SIGNAL: s(t) with noise

ANALYSIS: t-w domain

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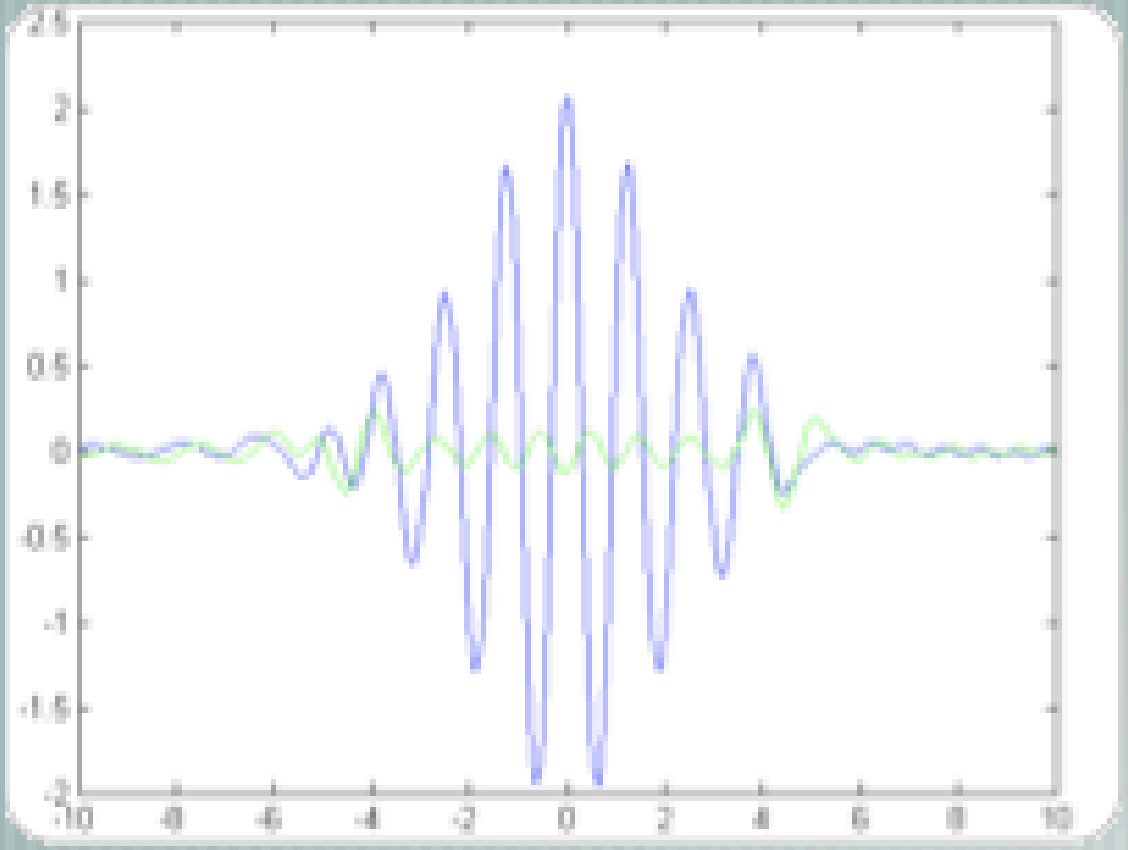
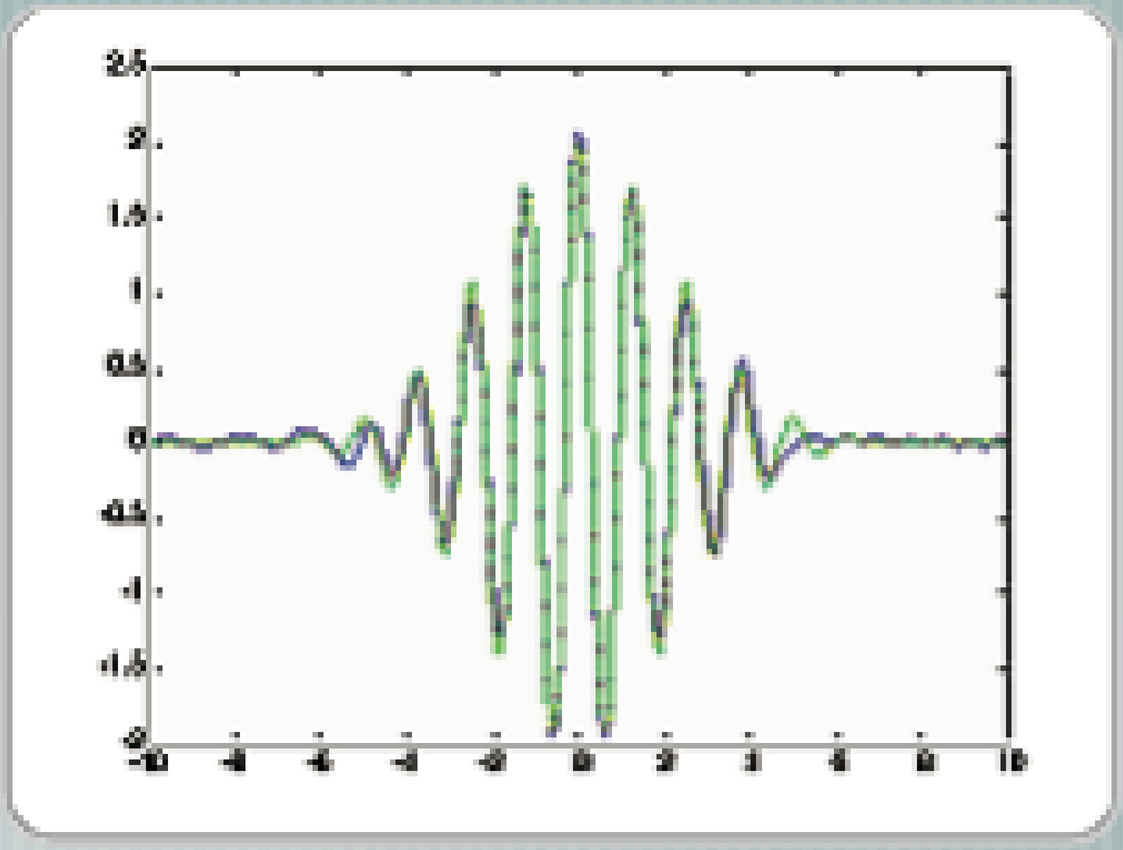
ANALYSIS: t-w domain

Filter Design part 2



SIGNAL: the output of filter

ANALYSIS: w domain



compare the output of the filter with the signal

ANALYSIS: t-w domain

Conclusion

- We have illustrated the effects of the FRFT / LCT and the effects of the FRFT / LCT operations.
- One of the application is S-method, finding the optimal angle and in this particular domain is the most particular one.
- Using the STFT and LCT to design a filter, the performance is better than FT one.

Future Work

1. Improve the design filter in more efficiency way.

2. Find out more powerful tool without the cross-term problem to do the time-frequency analysis.

3. Look for more application of FRFT and LCT.

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